

ON THE GRÜSS INEQUALITY FOR UNITAL 2-POSITIVE LINEAR MAPS.

SRIRAM BALASUBRAMANIAN

ABSTRACT. In a recent work, Moslehian and Rajić have shown that the Grüss inequality holds for unital n -positive linear maps $\phi : \mathcal{A} \rightarrow B(H)$, where \mathcal{A} is a unital C^* -algebra and H is a Hilbert space, if $n \geq 3$. They also demonstrate that the inequality fails to hold, in general, if $n = 1$ and question whether the inequality holds if $n = 2$. In this article, we provide an affirmative answer to this question.

1. INTRODUCTION

A classical theorem of Grüss (see [G]) states that if f and g are bounded real valued integrable functions on $[a, b]$ and $m_1 \leq f(x) \leq M_1$ and $m_2 \leq g(x) \leq M_2$ for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x)dx \right) \left(\int_a^b g(x)dx \right) \right| \leq \frac{1}{4}\alpha\beta,$$

where $\alpha = (M_1 - m_1)$ and $\beta = (M_2 - m_2)$.

A generalized operator version of the Grüss inequality was given by Renaud in [R], where he proved the following result.

Theorem 1. *Let $A, B \in B(H)$ and suppose that their numerical ranges are contained in disks of radii R and S respectively. If $T \in B(H)$ is a positive operator with $\text{Tr}(T) = 1$, where Tr stands for the trace, then*

$$|\text{Tr}(TAB) - \text{Tr}(TA)\text{Tr}(TB)| \leq 4RS.$$

If A, B are normal, then the constant 4 on the right hand side can be replaced by 1.

Among other operator versions of the Grüss inequality, of particular interest to us are those of Perić and Rajić (see [PR]), where they prove the Grüss inequality for completely bounded maps, and Moslehian and Rajić (see [MR]), where they prove the Grüss inequality for n -positive unital linear maps, for $n \geq 3$. In [MR], the authors show that the inequality fails to hold in general, if $n = 1$ and question whether it holds for the case $n = 2$. The main result of this article gives an affirmative answer to this question.

2010 *Mathematics Subject Classification.* 46L05, 47A63 (Primary), 47B65 (Secondary).
Key words and phrases. Grüss inequality, C^* -algebra, n -positive, completely positive.

Before we state the main result, we shall introduce some notation and definitions. Throughout this article, \mathcal{A} will denote a unital C^* -algebra, $M_n(\mathcal{A})$ the C^* -algebra of $n \times n$ matrices over \mathcal{A} , H and K complex Hilbert spaces and $B(H)$ the C^* -algebra of bounded operators on H . The notations $e, 1$ will denote the unit elements in \mathcal{A} and $B(H)$ respectively and $\phi : \mathcal{A} \rightarrow B(H)$, a unital linear map, i.e. a linear map such that $\phi(e) = 1$. The map ϕ is said to be positive if $\phi(a)$ is positive in $B(H)$ for all positive $a \in \mathcal{A}$. For more details, see [P]. It is easy to see that the map $\phi_n : M_n(\mathcal{A}) \rightarrow M_n(B(H))$ defined by $\phi_n((a_{ij})) = (\phi(a_{ij}))$ is unital and linear for each $n \in \mathbb{N}$. The map ϕ is said to be n -positive if ϕ_n is a positive map, completely positive if ϕ is n -positive for all $n \in \mathbb{N}$ and completely bounded if $\sup_{n \in \mathbb{N}} \|\phi_n\| < \infty$.

The main result of this article is the following.

Theorem 2. *Let \mathcal{A} be a C^* -algebra with unit e . If $\phi : \mathcal{A} \rightarrow B(H)$ is a unital 2-positive linear map, then*

$$(1) \quad \|\phi(ab) - \phi(a)\phi(b)\| \leq \left(\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left(\inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right).$$

for all $a, b \in \mathcal{A}$.

To prove Theorem 2, we use the well-known theorems of Stinespring, Russo-Dye, Fuglede-Putnam, and the result due to Choi (see Lemma 3).

2. PRELIMINARIES

In this section we include some lemmas which will be used in the sequel. Observe that if \mathcal{A} and \mathcal{B} are unital C^* -algebras and $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ is a unital n -positive linear map, then it is m -positive for all $m = 1, 2, \dots, n$. In particular γ is positive. It is well known that positive maps are $*$ -preserving. i.e. $\gamma(a^*) = \gamma(a)^*$ for all $a \in \mathcal{A}$. Moreover $\|\gamma\| = 1$.

Lemma 1. *If $P, Q, R \in B(H)$, then $A = \begin{pmatrix} P & R \\ R^* & Q \end{pmatrix} \succeq 0$ in $M_2(B(H))$ if and only if $P, Q \succeq 0$ and $|\langle Rx, y \rangle|^2 \leq \langle Px, x \rangle \langle Qy, y \rangle$, for all $x, y \in H$. Moreover, if $A \succeq 0$, then $\|R\|^2 \leq \|P\| \|Q\|$.*

Lemma 2. *Let $A = \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \in B(H \oplus K)$. If $R \in B(K)$ be invertible, then the following statements are equivalent.*

- (i) $A \succeq 0$
- (ii) $T, R \succeq 0$ and $T \succeq SR^{-1}S^*$.

The above two lemmas are well known. Their proofs can be found in [A].

Lemma 3 (Choi). *Let \mathcal{U} and \mathcal{V} be C^* -algebras and $\phi : \mathcal{U} \rightarrow \mathcal{V}$ be a positive linear map. If $x, y \in \mathcal{U}$ and $\begin{pmatrix} x & y \\ y^* & x \end{pmatrix} \succeq 0$, then $\begin{pmatrix} \phi(x) & \phi(y) \\ \phi(y^*) & \phi(x) \end{pmatrix} \succeq 0$.*

For a proof of Lemma 3, please see Corollary 4.4 of [C].

Proposition 1. *If \mathcal{B} is a unital C^* -algebra and $\phi : \mathcal{B} \rightarrow B(H)$ is a unital 2-positive linear map, then*

$$(2) \quad \|\phi(ab) - \phi(a)\phi(b)\|^2 \leq \|\phi(aa^*) - \phi(a)\phi(a)^*\| \|\phi(b^*b) - \phi(b)^*\phi(b)\|,$$

for all unitaries $a, b \in \mathcal{B}$.

Proof. Since ϕ is positive, recall that $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{B}$. Let $a, b \in \mathcal{B}$ be unitary. Consider the matrix

$$A = \left(\begin{array}{ccc|c} a^*a & a^*b & a^* & a^*(a^*b) \\ b^*a & b^*b & b^* & b^*(a^*b) \\ \hline a & b & a^*a & a^*b \\ (b^*a)a & (b^*a)b & b^*a & b^*b \end{array} \right).$$

Since a, b are unitaries, it follows that $R = b^*b = e$ and

$$T = \begin{pmatrix} a^*a & a^*b & a^* \\ b^*a & b^*b & b^* \\ a & b & a^*a \end{pmatrix} = \begin{pmatrix} a^*(a^*b) \\ b^*(a^*b) \\ a^*b \end{pmatrix} \begin{pmatrix} (b^*a)a & (b^*a)b & b^*a \end{pmatrix} = SS^* = SR^{-1}S^*.$$

Thus Lemma 2 implies that $A \succeq 0$. This is equivalent to

$$(3) \quad \left(\begin{array}{cc|cc} a^*a & a^*b & a^* & a^*(a^*b) \\ b^*a & b^*b & b^* & b^*(a^*b) \\ \hline a & b & a^*a & a^*b \\ (b^*a)a & (b^*a)b & b^*a & b^*b \end{array} \right) \succeq 0.$$

By Lemma 3 applied to the unital positive map ϕ_2 and the 2×2 block matrix in equation (3), it follows that

$$(4) \quad \left(\begin{array}{cccc} \phi(a^*a) & \phi(a^*b) & \phi(a)^* & \phi(a^*(a^*b)) \\ \phi(b^*a) & \phi(b^*b) & \phi(b)^* & \phi(b^*(a^*b)) \\ \phi(a) & \phi(b) & \phi(a^*a) & \phi(a^*b) \\ \phi((b^*a)a) & \phi((b^*a)b) & \phi(b^*a) & \phi(b^*b) \end{array} \right) \succeq 0.$$

This in turn implies that

$$(5) \quad \begin{pmatrix} \phi(a^*a) & \phi(a^*b) & \phi(a)^* \\ \phi(b^*a) & \phi(b^*b) & \phi(b)^* \\ \phi(a) & \phi(b) & \phi(a^*a) \end{pmatrix} \succeq 0.$$

By Lemma 2 and the fact that $\phi(a^*a) = \phi(e) = 1$, equation (5) is equivalent to

$$(6) \quad \begin{pmatrix} \phi(a^*a) & \phi(a^*b) \\ \phi(b^*a) & \phi(b^*b) \end{pmatrix} - \begin{pmatrix} \phi(a)^* \\ \phi(b)^* \end{pmatrix} \begin{pmatrix} \phi(a) & \phi(b) \end{pmatrix} \succeq 0,$$

i.e.

$$(7) \quad \begin{pmatrix} \phi(a^*a) - \phi(a)^*\phi(a) & \phi(a^*b) - \phi(a)^*\phi(b) \\ \phi(b^*a) - \phi(b)^*\phi(a) & \phi(b^*b) - \phi(b)^*\phi(b) \end{pmatrix} \succeq 0.$$

An application of Lemma 1 to the operator matrix in equation (7) yields

$$(8) \quad \|\phi(a^*b) - \phi(a)^*\phi(b)\|^2 \leq \|\phi(a^*a) - \phi(a)^*\phi(a)\| \|\phi(b^*b) - \phi(b)^*\phi(b)\|$$

for all unitaries $a, b \in \mathcal{B}$. Replacing a by a^* in (8) completes the proof. \square

The following three theorems are well known.

Theorem 3 (Fuglede-Putnam). *Let \mathcal{A} be a C^* -algebra. If $x, y \in \mathcal{A}$ are such that x is normal and $xy = yx$, then $x^*y = yx^*$.*

For more on the Fuglede-Putnam theorem, please see [B].

Theorem 4 (Stinespring's Dilation Theorem). *If \mathcal{B} is a unital C^* -algebra and $\phi : \mathcal{B} \rightarrow B(H)$ is a unital completely positive map, then there exist a Hilbert space K , an isometry $v : H \rightarrow K$ and a unital $*$ -homomorphism $\pi : \mathcal{B} \rightarrow B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in \mathcal{B}$.*

For a proof of Stinespring's dilation theorem, please see [P].

Theorem 5 (Russo-Dye). *Let \mathcal{A} be a unital C^* -algebra. If $a \in \mathcal{A}$ is such that $\|a\| < 1$, then a is a convex combination of unitary elements in \mathcal{A} .*

For a proof and more on the Russo-Dye theorem, please see [B].

3. THE PROOF

This section contains the proof of our main result, i.e. Theorem 2. The following theorem and corollary lead us to it.

Theorem 6. *If a, b are commuting normal elements in the unital C^* -algebra \mathcal{A} and $\phi : \mathcal{A} \rightarrow B(H)$ is a unital positive linear map, then*

$$(9) \quad \|\phi(ab) - \phi(a)\phi(b)\| \leq \left(\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left(\inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right),$$

i.e. the Grüss inequality holds for such $a, b \in \mathcal{A}$.

Proof. The proof is adapted from [PR]. Let $\lambda, \mu \in \mathbb{C}$. Since a, b are commuting normal elements in the C^* -algebra \mathcal{A} , it follows from Theorem 3 that the C^* -subalgebra of \mathcal{A} , say \mathcal{B} , generated by a, b and e is commutative. Since the restricted map $\phi : \mathcal{B} \rightarrow B(H)$ is positive and \mathcal{B} is commutative, it follows that $\phi : \mathcal{B} \rightarrow B(H)$ is in fact completely positive (see e.g. [P]). By Theorem 4, it follows that there exist a Hilbert space K , an isometry $v : H \rightarrow K$ and a unital $*$ -homomorphism $\pi : \mathcal{B} \rightarrow B(K)$ such that $\phi(x) = v^*\pi(x)v$ for all $x \in \mathcal{B}$. Since π is a unital $*$ -homomorphism, it is completely positive and hence is a complete contraction. In particular $\|\pi\| \leq 1$. It follows that

$$\begin{aligned} \|\phi(ab) - \phi(a)\phi(b)\| &= \|\phi((a - \lambda e)(b - \mu e)) - \phi(a - \lambda e)\phi(b - \mu e)\| \\ &= \|v^*\pi((a - \lambda e)(b - \mu e))v - v^*\pi(a - \lambda e)v v^*\pi(b - \mu e)v\| \\ &= \|v^*\pi(a - \lambda e)\pi(b - \mu e)v - v^*\pi(a - \lambda e)v v^*\pi(b - \mu e)v\| \\ &= \|v^*\pi(a - \lambda e)(1 - vv^*)\pi(b - \mu e)v\| \\ &\leq \|a - \lambda e\| \|1 - vv^*\| \|b - \mu e\| \\ &\leq \|a - \lambda e\| \|b - \mu e\|. \end{aligned}$$

The proof is complete by taking infimums on the above inequality first with respect to λ and then with respect to μ . \square

Remark 1. *It is easy to see that if \mathcal{A} is commutative or ϕ is completely positive, in the statement of Theorem 6, then the entire proof of Theorem 6 goes through with \mathcal{B} replaced by \mathcal{A} , for arbitrary a and b , i.e. the Grüss inequality (9) holds if \mathcal{A} is commutative or ϕ is completely positive.*

Corollary 1. *If ϕ and a are as in Theorem 6, then*

$$\|\phi(aa^*) - \phi(a)\phi(a)^*\| \leq \left(\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right)^2.$$

Proof. The proof follows by taking $b = a^*$ in Theorem 6. \square

Proof of Theorem 2. Recall a, b, \mathcal{A}, H and ϕ from the statement of Theorem 2. Let $\epsilon > 0$. By Theorem 5, there exist unitary elements u_1, \dots, u_k and v_1, \dots, v_ℓ in \mathcal{A} such that $\frac{a}{(\|a\| + \epsilon)} = \sum_{i=1}^k \alpha_i u_i$ and $\frac{b}{(\|b\| + \epsilon)} = \sum_{j=1}^\ell \beta_j v_j$, where $\alpha_i, \beta_j \geq 0$ and $\sum_{i=1}^k \alpha_i = \sum_{j=1}^\ell \beta_j = 1$. It follows from Proposition 1 and Corollary 1 that

$$\begin{aligned} & \frac{1}{(\|a\| + \epsilon)} \frac{1}{(\|b\| + \epsilon)} \|\phi(ab) - \phi(a)\phi(b)\| \\ &= \left\| \phi \left(\left(\sum_{i=1}^k \alpha_i u_i \right) \left(\sum_{j=1}^\ell \beta_j v_j \right) \right) - \phi \left(\sum_{i=1}^k \alpha_i u_i \right) \phi \left(\sum_{j=1}^\ell \beta_j v_j \right) \right\| \\ (10) \quad & \leq \sum_{i=1}^k \sum_{j=1}^\ell \alpha_i \beta_j \|\phi(u_i v_j) - \phi(u_i)\phi(v_j)\| \\ (11) \quad & \leq \sum_{i=1}^k \sum_{j=1}^\ell \alpha_i \beta_j \|\phi(u_i u_i^*) - \phi(u_i)\phi(u_i)^*\|^{\frac{1}{2}} \|\phi(v_j^* v_j) - \phi(v_j)^* \phi(v_j)\|^{\frac{1}{2}} \\ & \leq \sum_{i=1}^k \sum_{j=1}^\ell \alpha_i \beta_j \left(\inf_{\lambda \in \mathbb{C}} \|u_i - \lambda e\| \right) \left(\inf_{\mu \in \mathbb{C}} \|v_j - \mu e\| \right) \\ & \leq \sum_{i=1}^k \sum_{j=1}^\ell \alpha_i \beta_j \|u_i\| \|v_j\| \\ & = \sum_{i=1}^k \sum_{j=1}^\ell \alpha_i \beta_j \\ & = \left(\sum_{i=1}^k \alpha_i \right) \left(\sum_{j=1}^\ell \beta_j \right) \\ & = 1. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ above yields,

$$(12) \quad \|\phi(ab) - \phi(a)\phi(b)\| \leq \|a\| \|b\|.$$

Let $\lambda, \mu \in \mathbb{C}$ be arbitrary. It follows from equation (12) that

$$\begin{aligned} \|\phi(ab) - \phi(a)\phi(b)\| &= \|\phi((a - \lambda e)(b - \mu e)) - \phi(a - \lambda e)\phi(b - \mu e)\| \\ &\leq \|(a - \lambda e)\| \|(b - \mu e)\|. \end{aligned}$$

Taking infimums in the above inequality, first with respect to λ and then with respect to μ completes the proof. \square

The Grüss inequality fails, in general, when ϕ in Theorem 2 is assumed only to be positive, i.e. when $n = 1$, as the following example shows. We point out that [MR] contains an example of such a map $\phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$.

Example: Let $k \geq 2$, $\beta = \{e_1, e_2, \dots, e_k\}$ be an orthonormal set in H , $E = \text{span}(\beta)$, and $\theta : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$ denote the transpose map. It is well known that θ is a unital positive linear map, which is not 2-positive (see [TT]). Define $\phi : M_k(\mathbb{C}) \rightarrow B(H)$ by $\phi(a) = \begin{pmatrix} \theta(a) & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$. The block structure is with respect to the orthogonal decomposition $E \oplus E^\perp$ of H . Here $\mathbf{1}$ denotes the identity operator and $\mathbf{0}$ denotes the zero operator. It is easy to see that ϕ is a unital positive linear map which is not 2-positive. Let $a = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix} \oplus \mathbf{0}_{k-2} \in M_k(\mathbb{C})$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \oplus \mathbf{0}_{k-2} \in M_k(\mathbb{C})$. A simple computation shows that the eigenvalues of a belong to $\{0, 2 \pm \sqrt{10}\}$ and those of b belong to $\{0, 1, 3\}$. Since a and b are normal, it follows from [S] that,

$$(13) \quad \inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| = \sqrt{10} \quad \text{and} \quad \inf_{\mu \in \mathbb{C}} \|b - \mu e\| = \frac{3}{2}.$$

Moreover

$$\begin{aligned} \phi(ab) - \phi(a)\phi(b) &= \left(\left(\begin{pmatrix} 1 & 3 \\ 9 & 9 \end{pmatrix} \oplus \mathbf{0}_{k-2} \right) \oplus \mathbf{1} \right) - \left(\left(\begin{pmatrix} 1 & 9 \\ 3 & 9 \end{pmatrix} \oplus \mathbf{0}_{k-2} \right) \oplus \mathbf{1} \right) \\ &= \left(\left(\begin{pmatrix} 0 & -6 \\ 6 & 0 \end{pmatrix} \oplus \mathbf{0}_{k-2} \right) \oplus \mathbf{0} \right). \end{aligned}$$

Thus,

$$\|\phi(ab) - \phi(a)\phi(b)\| = 6 > \sqrt{10} \cdot \frac{3}{2} = \left(\inf_{\lambda \in \mathbb{C}} \|a - \lambda e\| \right) \left(\inf_{\mu \in \mathbb{C}} \|b - \mu e\| \right).$$

Acknowledgements: I would like to thank Prof. Scott McCullough for some useful discussions and Krishanu Deyasi for his help with some MATLAB computations.

REFERENCES

- [A] T. Ando, Topics on Operator Inequalities, Division of Appl. Math., Research Institute of Applied Electricity, Hokkaido Univ., Sapporo, 1978.
- [B] B. Blackadar, Operator Algebras: Theory of C^* -Algebras and von Neumann Algebras, Encyclopaedia of Mathematical Sciences, Vol 122, 2006, ISBN 978-3-540-28517-5.

- [C] M.D. Choi, Some Assorted Inequalities for Positive linear maps on C^* -algebras, J. Oper. Th., 4 (1980) 271-285.
- [G] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx$, Math. Z. 39 (1934), 215-226.
- [MR] M.S. Moslehian, R. Rajić, A Grüss inequality for n -positive linear maps, Linear Algebra Appl. 433 (2010), 1555-1560.
- [P] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge University Press, 1st edition, 2003.
- [PR] I. Perić, R. Rajić, Grüss inequality for completely bounded maps, Linear Algebra Appl. 390 (2004), 287-292.
- [R] P.F. Renaud, A matrix formulation of Grüss inequality, Linear Algebra Appl. 335 (2001) 95-100.
- [S] J.G. Stampfli, The norm of a derivation, Pacific J. Math. 33 (1970) 737-747.
- [TT] T. Takasaki, J. Tomiyama, On the geometry of positive maps in matrix algebras, Math. Z. 184 (1983), 101-108.

DEPARTMENT OF MATHEMATICS, IIT MADRAS, CHENNAI - 600036, INDIA.
E-mail address: `bsriram@iitm.ac.in`, `bsriram80@yahoo.co.in`